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Mathematics News Letter

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To mathematics in general, to the following causes in particular is this journal dedicated: (1) the common problems of grade, high school and college mathematics teaching, (2) the disciplines of mathematics, (3) the promotion of M. A. of A. and N. C. of T. of M. projects.

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A SINGULAR CONTRADICTION

We heard the dean of a well-known graduate school remark in a written address: Our universities are becoming veritable Noah's arks, since preparation for nearly every kind of trade or profession in the world is seeking to be fostered and promoted within their curricula.

Hearing this statement by one who, by the nature of his special work, must be expert in observing and interpreting the trend of higher educational currents, we were led to some reflections upon the present and possible future of mathematics in view of this form of university and college expansion.

Fifty years or more ago mathematics and the classics had an almost basic, at least a fundamental, place in the average college curriculum. This was the time when notions of culture and mental discipline weighed far more with school and college administrations than they now do. Since then there have been set into operation, at least within the United States, certain

educational doctrines which have had the effect of profoundly modifying the position of these two subjects of study in both the secondary and the advanced schools. We refer especially to the now almost universalized practice of election of subjects by both high school and college students-a practice largely due to the influence of President Eliot, of Harvard-; and steady attacks that have for years been made by certain psychologists on the essential values of the so-called formal disciplines of a given subject of study. Perhaps the latter part of this statement should be put somewhat differently. Efforts of pyschologists have latterly been directed to a precise and scientific measurement of the amount of training which the average student may be expected to carry over from one field of study into another not closely allied to it. Thus, while such experimentations may truthfully be said to have implied no "attack" upon the disciplinary merits of mathematical study, undoubtedly the effect has been to inject doubt into administrative minds which previously had harbored none as to the power of mathematics, properly studied, to sharpen the logical and analytical faculties for exercise in any field whatsoever.

Alongside of these two influences which have served to partially change the original rank of mathematics in the schools has gone another influence which, however unconsciously, has operated in the same direction. We have reference to the steadily increasing practice of the States of placing upon their own colleges and institutions the burden of educationally caring for all their youth, regardless of the utter incapacity of large numbers of them for assimilating a higher education.

We do not need to demonstrate to the reader that these influences have been and are at work. They are generally recognized facts. A sample evidence of the changed situation in reference to the place of mathematics is afforded by a certain modern university in these figures, taken from a recent catalogue. Teaching mathematics, 24, teaching zoology, 23, teaching history, 39, teaching physics, 38, teaching chemistry, 57. Put otherwise, in this great university are found 95 teachers of chemistry and physics, sciences requiring more or less of mathematics as a basic preparation for their study, while only 24 teachers are employed to teach the mathematics itself.

The "singular contradiction" referred to in the heading is this: Keeping equal pace with these movements which have slowly brought about a relative displacement of mathematics from its original primacy in the school curriculum, remarkable to say, has been **a steady expansion of the demand for a knowledge of mathematics as a prerequisite to successful work in fields of industry supposedly unrelated to mathematics.** It is but natural that the lay mind should not know that this is true. That it is, never-the-less, true can be attested by scores of industrial and technical leaders-those who are not professional mathematicians, nor professional boosters of mathematics.

A bright young instructor was about to leave for a university in the west to work for a doctor's degree in education. Before leaving he said, "The particular line of my studies will require familiarity with a certain type of mathematics. This familiarity I am lacking. I have misgivings because of it". A young woman, university graduate, successful high school teacher, said to us: "The further I try to go in the physics field, the more hadicapped I become because of my little mathematical knowledge. I must take more mathematics". A mature student of agriculture with doctor's degree from a German University found that his research problem demanded a knowledge of certain mathematics . A friend asked, what will you do about? He answered: "I shall go back to the university and get the mathematics."

By what broad path shall a quasi-science become a true science? The answer is: By the path mathematical, for the accurate formulation of the laws of any science must in the end be a mathematical formulation.

So, we conclude where we started: It is a singular paradox in the world situation to-day that along with the schools' steadily relaxing grip on the most basic of sciences is going a steadily increasing demand by industry and science for a knowledge of that same basic science, MATHEMATICS. —*S. T. S.*

MATHEMATICS NOT FOR THE GIFTED ONLY

There are certain types of mind whose instinctive urge is towards—not the easiest—but the most difficult problem. Often

the urge amounts to an insatiable appetite. These minds are not found attacking questions manifestly unsolvable, as, for example, the origin of matter, the existence of life on other planets, or the nature of electricity. Neither, on the other hand, as a rule, are they over-concerned about so trite a problem as making a living or even a fortune. It is within the domain mathematical that most of them find their greatest satisfaction.

But because of the almost infinite variety of materials which compose mathematics not only may the the most gifted intellect find adequate exercise in its study but in equal degree may minds of ordinary ability, by proper selection from its offerings get rich returns in culture and power. Its phases are numerous. There is the **measure** phase, the **computation** phase, the **function** phase, the **geometric** phase, the **number** phase, the **group** phase, the **logic** phase, the **philosophy** phase, the **applied** phase—and other phases. In a sense it is the abstract of all science. The outlook of its processes is in so many directions, its points of contact with life and reality are so numerous, that no other subject is as prolific in the yield of human interest. It is in the face of this fact that one's wonder grows—indeed, one is almost startled—when one is told that teachers are failing in their efforts to tap the wells of student interest in the subject, that, as a rule, it is let alone, when possible, by young people under their privilege of election. **Historic** and **biographic** phases alone are sufficient to invest the study of any mathematical topic with a thrill for the young mind—unless it is dumb or dead to begin with! Archimedes, dashing, clotheless from his bath, shouting, "Eureka," to note down for the world his discovery of the principle of specific gravity; the lost Greater Fermat Theorem, the struggle of a mathematical world to rediscover it, the consequent creation of new realms of number theory; the personal meeting between Briggs and Napier, inventors of the two systems of logarithms in world-use, when each sat 15 minutes admiring the other in silence before speaking: these incidents reveal a superlative humanness of struggle of mind as intense and red-blooded as the struggle of armies bent on conquest. And, best of all, the number of such very human narratives about mathematics and mathematical workers is almost without limit! But the TEACHER must be all there! —*S. T. S.*

QUESTIONS AND OPINIONS

By W. PAUL WEBBER
Louiiana State University.

Some time ago I received a questionnaire on the Teaching of Geometry. I did not find time to answer it so as to do the questioner any service. It may be interesting to consider it here. If this should ever come to the eye of the questioner, I hereby make apologies for my earlier negligence. The questionnaire method has been popular for some time. It is not so popular now as formerly. After all what does one get by this method but opinions. These may have a use, but they are not science. In the data gathering stage opinions may be valuable. What I shall say will be opinion. Do not take it too seriously. After each item I am appending my comment.

Under the heading "Aims or Objectives."

1. Appreciation of geometric form in nature, art, and industry. Now I ask how many, actually and consciously, have these in mind while teaching geometry? It is worth considering as a motive, also. Do teachers generally have these objectives in mind?

2. Knowledge of facts and principles of geometry.

I am of the opinion that this objective holds an important place for most teachers.

3. Knowledge of the contribution of geometry to civilization.

I am skeptical about this objective having much attention in average teaching.

4. To develop space perception and imagination.

Is enough attention given to this or should this be done in earlier grades?

5. To teach the meaning of a proof or a demonstration.

This is one of the classic claims and is valid. But is it an exclusive possession of geometry? Does the geometric form of proof give a sufficiently broad idea of a proof?

6. To develop the power of clear thinking and precise expression.

Are these appendages to the foregoing objective?

7. To develop the ability to analyze a complex situation into simpler parts.

Is much of this done, consciously, by teachers?

8. To develop the inquiring or questioning attitude of mind.

Does it? Can any isolated school branch do much of that? Is it not more an atmosphere in civilization?

9. To develop the ability to subject a statement to a severe test of its validity.

Is this a peculiar function of geometry? To what extent do you attend to this in your teaching?

10. Learn model proofs and develop memory?

This should need no comment. Proofs are memorized, but are models learned? Is there a difference between memorizing a proof and learning it as a model?

11. To meet college entrance requirements.

A few colleges may require entrance examinations.

12. To prepare for the study of science and higher mathematics.

Would not a more thorough study of half the present content, spread differently, improve the situation?

13. To solve originals and develop the power of invention.

How many get this to any extent?

14. To make clear the process of deductive thinking.

How much does this overlap (10)?

15. To train in functional thinking by means of geometric relationships.

This is possible but how many get it. Most of my college students are not consciously aware of The Pythagorean Theorem as an instrument in thinking.

16. To make applications in drawing to scale.

Can this be accomplished along with all else?

17. To give pleasure and mental uplift by contact with exact truth.

Why associate "mental uplift" with pleasure? There is usually some pleasure in the study of geometry unless the teacher is a kill-joy or the supervisor buts in with too many "yes and no" tests.

19. To develop mental habits and attitudes that are needed in life situations.

Now just what does this mean in this form. Unless something definite is specified this is only mystery mongering.

20. To develop moral and spiritual ideals.

Same comment as on (19).

I have been guilty for, many years, of advocating these objectives and others. I believe in some of them. They should be useful in producing good citizenship and personal efficiency.

It will be noticed that considerable importance is attached to the types of reasoning developed in geometry. Of course, the questionnaire does not contemplate anything but geometry. It might be of interest to mention some of the types of demonstration used in algebra. In freshman mathematics as at present constituted there is less of the standard geometric style of procedure than in geometry as presented in high school.

1. Suppose it has been shown that a quadratic equation has two roots. What reasoning is used to show that it has no more than two? As near as I can make out from the reactions of students it is mere habit or long experience with the vast majority. This would be pure induction such as the physicist or biologist uses. Can it not be done in another way quite simply?

2. What type of reasoning is used in establishing the laws of permutations and combinations? The same inductive type. The same is true with the binomial theorem. For the next step, called mathematical induction, is beyond the reach of pupils of this grade and to enforce it on them is only to make them think it is absurd and make most of them hate it.

3. What type of reasoning is employed in generalizing the ordinary rules of factoring? What do you suppose your pupils would do if asked to give a proof that a^n+b^n is exactly divisible by $a+b$ if n is odd?

The majority of my pupils will give an illustrative example, unless I caution them. Then only a few will answer the question at all.

I might go on. But I will ask one or two more questions. In teaching do we give enough attention to making pupils conscious of the various types of reasoning or are we satisfied if

they learn the proof in the book with little or no comment. Are we using algebra and geometry as educational instruments, or are we just teaching algebra and geometry, trusting the good Lord for the educating phase of it? Is it possible to do more.

As a teacher of mathematics do you believe that English can be taught so as to bring out the different types of reasoning? Think of it! If this is possible may we not have to look to our laurels to maintain our claims. Now, I believe in the claims if the teaching is good. I believe mathematics has utilitarian values today that it never had before, but as experts real or pretended, have we a duty to perform?

MATHEMATICS AT EXETER

By HARRY C. BARBER
Head of Mathematics Department, Exeter Academy
Former President, National Council of Mathematics Teachers

(Through the kind permission of Professor Barber we offer this reprint of one of his papers from the Bulletin of the Phillips-Exeter Academy—the September, 1929 issue. When it is remembered that this academy has had a history covering one hundred fifty years of distinguished and effective work, that some of the greatest men in America nourish it as their alma mater, that its mathematical department, through Professor Wentworth, did more to develop the proper teaching of mathematics in America, than any any other single agency, there is no doubt that readers of the Mathematics News Letter will be stimulated by reading the whole of Professor Barber's paper-the paper of one who by general acclamation has been a worthy successor to Professor Wentworth. —S. T. S.)

"I was taught here to learn." These words and those of the parent who says, "You taught my son to work," contain the tribute perhaps most commonly paid to the Academy. What is meant is that the drive of the place has led the boy to acquire habits of industry and persistence, to assume responsibility for his own success, and to refuse to be beaten by hard tasks. Realizing that he had to compete in order to get in to the school and that he had to struggle in order to remain, he readily accepted the hardwork tradition of a school in which "the student bears the laboring oar."

This drive and the boy's response to it is a characteristic of the Academy which must be understood before the work of any department can adequately be discussed. Two other general characteristics should be mentioned. This is a school for able boys. Modern education increases its efficiency by grouping students according to their ability. At Exeter this separation automatically occurs because boys from the lower ranges of ability are very generally excluded. Here the intelligent boy competes with his equals, and methods of instruction can be adapted to his capacities and needs. And finally there is success in examination for college entrance. Few failures occur and high averages are maintained by those who pass their courses in the Academy.

It is obvious that these three characteristics of the Academy constitute a challenge to each department. The pact must be maintained. It is, on the other hand, just as obvious that they do not in themselves constitute a sufficient scholastic aim. None of the departments would be content with plans which did not contemplate more than this. All of them are accustomed to say, in effect, that they expect not only to send boys on into college but to send them there with eager intellectual interests and with the capacity and will to grow. Now it is clear that such an achievement is not entirely a matter of specific training, of information, and of petty "skills;" rather it has to do with attitude of mind. It may be that this is the true measure of success; that excellence in a school lies in the attitudes of mind which it creates.

The more one ponders the problem the more convincing this conclusion becomes. The attitude is the thing. Success and failure depend upon it. It determines all the adjustments of our lives. One educated man is a criminal, one an irresponsible idler, and a third a benefactor; it is a matter of attitude of mind.

This conclusion is not only sound but far-reaching in its effects, for when we accept as the measure of our success the mental attitudes which we develop in our students, it follows that every method and device and the whole spirit of the institution will be determined with regard to the attitudes which we develop in our students, it follows that every method and device and the whole spirit of the institution will be determined with

regard to the attitude of mind which they produce. With this conclusion in mind we turn to the discussion of what the department of mathematics hopes to do.

I

Many of the traditions of the department are associated with Wentworth, of masterful personality, whose textbooks made his name a household word throughout the country. He helped to shape school algebra into a well defined unit of work. He was a leader in the reform that put originals into goemetry. How well Mr. Francis caried on the work is not so widely known, but it is clearly indicated by the opinions expressed by his students who have been long enough out of school to see their education in perspective. Two alumni, one himself a mathematics teacher of long standing, and the other a business man well versed in mathematics, have told me recently not only of their admiration for the man and his devotion to the Academy, but of the unusually sane and stimulating view he gave them of mathematics, its nature and its purposes. Such voluntary tribute is not easily won. These two teachers have left us the task of continuing the progress they began.

Wentworth was himself the whole department. Now we pool the wisdom and experience of seven men co-operating for the improvement of instruction. To bring about this mutual study and co-operation was one of the problems resulting from the growth of the school, and it is interesting to see that the traditional democracy of the Exeter faculty lays the right foundation for the necessary free interchange of ideas. In other words, during the present period of growth of American schools there is no more important movement than that looking to the improvement of instruction through co-operative endeavor and discussion on the part of the teachers, and to this plan of co-operation the free give and take of the faculty meeting and the general traditions of the school are notably well adapted.

In the effort to maintain past excellence and to adapt it to modern times we know that we must move ahead. We recognize the difficulty of weighing the old against the new, of making changes without loss. We know too that our subject is so old and has been so long a part of the school curriculum that

some would think no changes in it need be made. Yet changes are being made, so many in fact that we now commonly hear the phrase, "the new mathematics." There is of course no new mathematics in the field of the secondary school, but two things are taking place: the same two that took place in Wentworth's day: I. The usual periodic revision of aims, material, and method. II. The perennial struggle of good teachers, each in his day and generation, to resist the bad trends and to support the good. The preface of a mathematics text of one hundred years ago boasts that pounds, shillings, and pence are replaced by decimal coinage in all its problems, and that its students are taught to think and to depend upon themselves. Prefaces today make not dissimilar claims.

We have in mind both this revision and this upward struggle, and they may be said to give color to much that is distinctive in mathematics at Exeter. It so happens that school mathematics falls easy prey to certain ills. They are suggested by such words as rote, routine, drill for drill's sake, empty symbols, and meagre aims. This is particularly true, as has been pointed out again and again, in schools which prepare for external examinations. The mere mastery of a process so that it can be reproduced tends to be magnified over the intellectual curiosity which may be aroused, the methods of investigation which may be taught, and the direction which may be given to thinking by the proper approach to these processes and to the purposes which they serve. Resistance to the commonly recognized belittling tendencies of the mathematics classroom is one of the marks of good teaching. We try to have in mind always that "overhaste to impart mere knowledge defeats itself." Even the so-called cramming school can fix the processes. It is the margin which we get over and above the processes themselves by which we want to stand or fall.

II

This brings us back to our first conclusion, that excellence in a school or in a department lies in the attitudes of mind which it creates. There are many mental attitudes which we may hope to develop. Some of them are suggested in the paragraph above. Perhaps the most interesting and the most important

one for our department is that which may be called the scientific attitude of mind. The present age is pre-eminently scientific. It is set apart from all other ages by scientific discovery and invention which are due to the scientific method and the scientific attitude of mind; and thoughtful men are already predicting that the outstanding characteristic of the coming generation is likely to be the application of this method to more and more of man's problems with more and more success. President Eliot wrote, "The nineteenth century brought into the world for the service of education, as well as for the service of industries and government, the new temper of mind called the scientific; and the effects of this new temper or spirit have been nothing less than revolutionary. It is surely true that in order to understand the genius of our own age we must know something of the scientific method of thought. It is probably true that in the preparation of our students for the age in which they are to live we can perform no greater service than to make them familiar with this method and its use. We may even believe that in this complex age the straight, systematic, and unprejudiced thinking that is characteristic of the scientific habit of mind is the necessary foundation for the building of intelligent character.

Now mathematics offers the earliest and best opportunity to acquire the scientific habit of mind. Of course it is true, as the reader is no doubt thinking, that the random study of the subject without regard to methods of approach or to the resulting attitude of mind may completely fail to accomplish this end. No one can safely claim that this is not so, but the opportunity is there and we believe that if we persistently seek the development of this attitude of mind, and if we use our material intelligently for the purpose, much can be accomplished.

III

One more mental attitude is deserving of our close attention. The question at issue is this: How much do we want the interest of the student in his work? Much, little, or not at all? So long as he does the hard work, need we be concerned about this interest in it? This question is fundamental because it determines so many details of the teaching methods to be used.

Life brings to nearly all of us difficult tasks. We know the value of learning to do hard work. Even the schoolboy does, at the bottom of his heart. This does not mean, however, that the secondary school must condemn him to four years of hard, uninteresting labor. Interest in his work does not make him less diligent, less successful, or less happy. Interest does more than anything else to develop that willingness and eagerness of mind which is fertile soil for education. Lack of genuine interest is probably the most common cause of failure.

Whatever can be accomplished by hard work under compulsion, all that and more can be accomplished by hard work impelled by interest. The galley slave bears the laboring oar under the threat of the lash, but the school crew labors just as hard because of interest in the outcome. Both kinds of rowers develop muscle, but the school boys probably develop the better attitude of mind. In life's lower ranges there is the compulsion of fear or hunger, but it is interest which leads to the painting of a great picture or the writing of a poem. It was interest which led Archimedes to double the knowledge of geometry which was handed down to him by Euclid.

This brings us back again to the heart of the matter. We must admit that when we undertake to prepare our students for college in such a way that they will have the desire for more education and the ability and determination to succeed, when we accept their attitude of mind as our major aim, we have an aim which cannot be attained by external pressure alone. The groundwork of success is the interest of the students, and the larger the proportion of them who work hard because of their own interest the better we are succeeding. It is clear, of course, that we are not talking about idle curiosity or mere amusement but about interest of the sort which makes one want to get work and achieve something.

The scientific attitude and the attitude of interest rather than of yielding to compulsion are not without relation to each other. When the student realizes that "Mathematics underlies our present day civilization in much the same fundamental way that sunshine forms the source of all life and activity on the earth," and when he finds that he is learning to use the scientific

method of thought, he is very likely to develop an interest in his daily work. As he progresses through one, two, three, or four years of such study of mathematics he may be excepted to show noticeable growth year by year not only in knowledge but also in that understanding and questioning attitude of mind which may properly be called education.

A DISCUSSION

By H. L. SMITH
Louisiana State University.

In the last issue of the News Letter (April, 1930), Professor W. Paul Webber proposed for discussion the question, "At what point in teaching first year college mathematics should the idea of a general variable be introduced in trigonometry?" The writer has very definite notions on this point and it is the object of this note to set forth these notions.

Trigonometry falls naturally into two parts: (1) numerical trigonometry, and (2) analytic trigonometry. The first of these is devoted to the solution of triangles and this object gives a high degree of unity to it. The second part is devoted to the study of the circular functions and their inverses and is chiefly needed as a preparation for the calculus.

Now numerical trigonometriy is based upon the trigonometric functions of geometric angles. These should first be defined for acute angles and applied to the solution of right triangles. The definitions should then be extended to obtuse angles in such a way that the law of sines and the law of cosines are both true for all triangles. This leads to the following definitions of sin A, cos A when A is an obtuse angle.

(1) the sine of an obtuse angle is defined to be the sine of the supplementary acute angle;

(2) the cosine of an obtuse angle is definite to be the negative of the cosine of the supplementary acute angle.

The treatment sketched above of course requires that a geometric proof be given of the law of tangents.

Let us now consider analytic trigonometry, and recall that

it is devoted to the study of the circular functions of a numerical variable. We indicate the definitions of the functions sin t, cos t. On the circle $x^2+y^2=1$ set up a coordinate system with such a unit that the points (1, 0), (0, 1) have the coordinates $0+2n\pi$, $\pi/2+2n\pi$, respectively. We then define

$$\cos t=x, \sin t=y$$

where x, y are the coordinates in the plane of the point on the circle whose coordinate on the circle is t.

The definitions just given can be made very concrete and vivid to the student by means of a diagram in which the t-scale, x-scale and y-scale are actually drawn in considerable detail on coordinate paper. Moreover these definitions lead at once to a very simple and general treatment of the addition theorems on which the whole of analytic trigonometry is based.

The reader will by this time have inferred the writer's answer to Professor Webber's question. It is that the general variable should be introduced at the beginning of analytic trigonometry. Moreover we have indicated how that variable can be introduced in a very concrete way.

EUCLID AND INFINITY

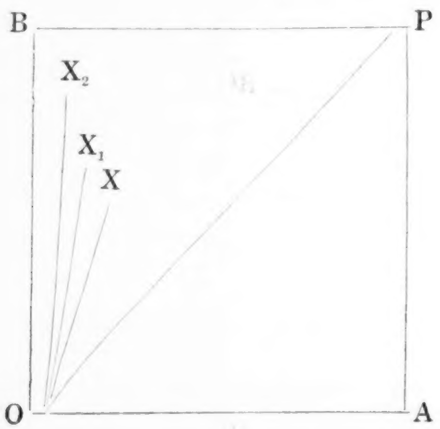
By S. T. SANDERS

Euclid's definition of parallels may not rightfully be asserted to imply that straight lines are infinite in length. When he says that two straight lines are parallel if, on being produced indefinitely, they do not meet, he is by no means postulating the existence of a point "at infinity" on either line. He merely implies that they have the property of being indefinitely extensible. The postulate of Archimedes expresses the matter with precision. Carslaw puts it thus: "If two segments are given there is always some multiple of the one which is greater than the other." It is one thing to imply that for every segment, however long, there exists one still longer, but quite another thing to assert that there exists one so long that it contains an infinitely distant point. To say that on a straight line is an infinitely distant point is equivalent to saying that there is an actual positive integer, namely, "infinity," greater than all other integers. All we assume in arithmetic, or number theory, is

that, however great may be a designated integer, there exists one still greater.

The distinction here elaborated is not intended to affect the quite convenient usage sometimes adopted by writers when they refer to Euclid's parallels as "meeting an infinity," a mode of expressing the fact that they do not meet at all.

It is our purpose in this short article to point out somewhat more clearly and specifically than the elementary books do some interesting consequences of the indefinite extensibility of a straight line.



Suppose that the straight lines OB and AP are perpendicular to the straight line OA. Let us suppose also that OP intersects AP as shown in the diagram. Finally let us suppose that AP is indefinitely extended so that the acute angle POA is made to approach in magnitude the right angle BOA. It is implied of course that OP also must suffer indefinite extension. A second implication is that, under the conditions prescribed, among which is that P must always be on both OP and PA, angle BOP must always be of magnitude greater than zero.

Manifestly the folowing conclusions may be drawn:

(1) However far AP and OP may be extended, within the angle POB may be drawn through O an indefinitely great number of straight lines none of which will be cut AP between points A and P, some of which, however, will cut AP if the latter be sufficiently extended.

(2) If OB and AP are kept always equal, the lines designated in (1) will be denumerably infinite since there is an infinity of points in the segment PB.

(3) Segment PB will always be equal to segment OA.

(4) Within the region BOP choose a straight line OX so that angle BOX is small at will. Holding OX to some fixed length and direction while AP is extended and angle POA enlarged, let the extension continue until OP arrives at a position of coincidence with OX. When this coincidence has been attained choose another line in the region BOX, namely OX_1. No point on OX_1 is also on AP. Furthermore, between OX_1 and OB is an infinity of lines not crossing AP. Again extend AP until OP coincides with OX_1. When this has been done choose another line in the interval BOX_1, namely OX_2. No point on OX_2 is on AP. Furthermore, between OX_2 and OB is an infinity of lines not crossing AP. It is evident that the process here described may be carried on indefinitely. However many times it be repeated, there will always be a line OX_n in the neighborhood of OB and lying between OB and AP which will have no point in common with AP, and which, with OP determines a region BOX_n within which an infinity of lines through O exist without cutting AP.

(5) The essence of (4) may be put in the following manner: No amount of extension of the parallels OB and AP will ever result in making angle BOP so small but that an infinite number of lines OX exist that do not intersect AP.

Remark: Those acquainted with non-euclidean geometry readily recognize in the statement of (5) one very closely allied to another postulate (at variance with Euclid's parallel postulate), namely, through a point not on a given straight line, more than one parallel can be drawn to that straight line.

(6) Assume any line segment within the region BOP, as OX, and let it be close at will to OB. By sufficient extension of AP line PO may be made to coincide with OX so that X will fall on OP and thus by extending OX if necessary, OX can be made to cut AP. Then assume another line OX_1 still closer to OB. By repeating the process just described for OX_1 the latter may also be made to cut AP. In similar manner any other line falling on line OB and passing through O, as OX_n, making

the sum of the angles AOX_n and PAO less than 2 right angles, can, if held fast and sufficiently extended, be made to meet AP if the latter is also properly extended. By being "held fast" is meant that its direction remains constant while AP is being extended. The conclusion would not be valid without this assumption of a constant direction for OX_n, for example, if it should assume a slight curvature to the left as it is being extended.

Remark: Students of Euclid recognize in the statements of (6), not a proof of Euclid's parallel postulate, for no such proof can be made, but certainly a ground for its assumption.

Infinity.

As a natural supplement to the above remarks on the "indefinite extension" properties of Euclidean straights, we offer the following on the notion "infinity."

No term in mathematics has a more indeterminate meaning than this one, yet none other has a more mystic attraction for the mere beginner in mathematical study. The freshman may forget that the cosine of O is 1 or that $\sin \pi = 0$ but never will he forget that the tanget of $\pi/2$ is infinity—whatever that may mean to him.

We are strongly inclined to believe that much of the use of ∞ as the symbol of a number—even by the best writers—has been the consequence of an instinctive attraction to the intellectually unknowable. This is our feeling, regardless of the fact that a considerable body of the mathematical literature constructed by such minds as Russell, Cantor and others has been built up on such subjects as "infinite cardinal numbers," "transfinite number systems," non-inductive numbers.

To one familiar with the quadratic equation there is crystal-clear meaning to the proposition: x has either of two values, if $x^2—5x—11=0$. Each of these two numbers is found by an absolutely definite process. Again, one with even slight acquaintance with analysis will make no mistake in interpreting the formula $2 \leqq x \leqq 5$. Nor is it necessary in order to take in its complete meaning to do so via the so-called "infinity" of values which x assumes within the prescribed interval. In truth when we speak of a variable "taking on" an infinite number of

values we must think of a conscious assignment of such values, which is an absurdity. The form means nothing more nor less than than x may have **any** value assigned to it which is greater than or equal to 2 and also less than or equal to 5. Again there is the same degree of unconditioned definiteness in an inequality such as $0<x$, x any real number, positive. Contemplation of "well orderedness," or "denumerably infinite" properties of the class of numbers "x" is unnecessary to clearly sense the statement's meaning. The "infinity" of numbers covered by the x symbol is an aspect of the statement no more essential to a grasp of its meaning than a knowledge of how many apples are on a tree is necessary before a basketful can be picked from it. Nor is ∞ (?) one of these values. Any operation to which x, as here defined, is subjected, yields a unique number—unique though expressed in terms of a number symbol of great generality. Let x be **any** positive number, then is $x^3—x—8$ also **some** number. So is any other function of x.

Suppose now, in contrast to the convention that x is to have **one** and **only** one of its infinitely many values when subjected to a mathematical operation, we allow it to connote many of these values. Then will every expression resulting from that operation have many values, or, put otherwise, all mathematical operations on x will no longer be unique. Now the very genius of a mathematical process is the uniqueness of its results. Bearing this in mind we undertake a next step in our analysis.

Let us try to interpret such an inequality as

$$0<x<\infty \qquad \text{(i)}$$

in which, as in the previous hypothesis, x is to represent any single real number (positive).

Our study will lose nothing by considering only the values of x which are positive integers. Since for every positive integer n which may be named there is one larger, namely, $n+1$, it follows that if the symbol ∞ is to represent a number in such a way as to make (i) function as a true inequality the number represented must be a kind of number which is not reached by any counting or enumeration process, that is, it must be greater than every integer, and hence greater than every real number. While manifestly, the assumption of the

existence of such a number may be made without logical inconsistency as, fo rexample, when we assume that ∞ is the number of points on a line segment, on the other hand the absence from such a number of those properties which are necessary in order that unique results follow from subjecting it to addition, subtraction, multiplication, division and other operations, renders it useless, except in special cases.

Justifying (i), then, as a logical convention, it becomes convenient to classify all numbers as belonging to one of the two classes, namely, the **finite,** or the **infinite** class, the symbol for a member of the latter class being ∞, or infinity. Thus, do we find cautious writers on analysis frequently using such language as, "let y be any finite number," "integrate between finite limits," "let n be a finite number, not zero," implying that they recognize the existence of numbers not finite.

It is one thing to construct a number, or a number system, but quite another thing to make effective use of it. It is mainly in connection with **limit** theory that ∞ has its practical uses in elementary mathematics. The examples which we now give, showing some of its legitimate uses and the limitations of its uses, may be of value to certain readers who find, as a rule, no completely organized body of ideas pertaining to it in the elementary literature.

Because ∞ is defined solely by the property that it is greater than all positive integers and hence is a number void of the inductive, or enumerative property, it follows that it must be unaffected by the addition to it of 1. In other words $\infty+1=\infty$, from which it easily follows that $\infty+k=\infty$, k being any ~~in~~finite number whatsoever, so that such statement as $\infty-\infty=k$ has, of itself, no determinate value.

Consider $1/x$ for values of x near to 0 at will. By assigning to x a sufficiently small value, $1/x$ can be made to become greater than a previously designated integer, however large. The nearer to zero is the value assigned to x the larger is the resulting number $1/x$, and as long as x is not equal to zero, so long is the quotient a definite finite number. Thus, consistent with the definition of ∞ as a number greater than all finite numbers is another assumption; namely, $1/0=\infty$. But by the same test $k/0=\infty$, k being any finite number. Thus must such product (?)

as $\infty \cdot 0$ be also indeterminate when regarded solely by itself.

Again, consistent with $k/0=\infty$, is $k/\infty=0$.

As k is any finite number, no previous assumption or deduction is contradicted if from

$$k_1/\infty=0 \qquad \text{and} \qquad k_2/\infty=0$$

we conclude

$$\infty/\infty=0/0.$$

Thus, as $0/0$ is known to be an indeterminate form, it follows that $\frac{\infty}{\infty}$ is also indeterminate, that is, regarded solely within itself.

The student of the calculus will recognize that all the above named indeterminate number symbols, that is $\infty-\infty$, $\infty\cdot 0$, $\frac{\infty}{\infty}$, are used by most writers as symbols for absolutely definite finite numbers in cases where such numbers appear as LIMITS of well-defined variables. This we now illustrate.

Consider the function $1-x^2/1-x$, and let us determine a possible LIMIT of its value while x assumes values that approach 1. Put otherwise, let us find a number, if possible, which is defined by the process

$$\text{Limit } (1-x^2/1-x \\ \text{as x approaches } 1$$

Since for every finite value of x diffeıent from 0, $1-x^2/1-x=1+x$, we must have

$$\underset{\text{as } x \text{ approaches } 1}{\text{Limit } (1-x^2/1-x)} = \underset{\text{as } x \text{ approaches } 1}{\text{Limit } (1+x)}$$

hence,

$$\underset{\text{as } x \text{ approaches } 1}{\text{Limit } (1-x^2/1-x)=2} \tag{1}$$

Putting $1-x^2/1-x$ into the three forms, namely,

(a), $(1-x^2)\ (1/1-x)$ (b), $(1/1-x)\ (1/1/1-x)$

(c), $((1/1-x\ -\ x^2/1-x)$

and permitting x to be replaced by zero in accoraance with the above assumptions concerning infinity, we have the three formally indeterminate symbols,

(a)′ $0\cdot\infty$, (b)′ $\frac{\infty}{\infty}$, (c)′ $\infty-\infty$ (2)

In the light of (1) and (2) there is no logical inconsistency in our arbitrarily associating any one of the symbols (a)′, (b)′,

(c)[1] with the finite limiting value, namely, 2, of the function $1—x^2/1—x$, as x approaches the value 1. It is in this sense only that use is made of them by the writers on analysis.

Concluding this phase of our study it should not be overlooked that each of the three infinity symbols of (2) is convertible into an equivalent zero-symbol, namely 0/0, a fact which makes unnecessary the use of the former symbols.

We gravely question the advisability of allowing a freshman to use the term infinity even when all the delimitations of its meaning are first put before him in some such manner as above. There is more than enough for him to unravel and make consistent when his studies are restricted to the finite numbers. He should not be needlessly mystified. It is better, far better, to drill into him that division by zero is not a valid operation. Train him to say, "as $\angle x$ approaches $\pi/2$, the tangent of $\angle x$ grows large without limit." This mode of statement is definite, and, intuitively, at least, its truth can ultimately, if not immediately, be made clear to him by its use.

ON PHYSICALLY LARGE AND SMALL QUANTITIES

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Some of the quantities we meet in the study of Physics and Astronomy are so extraordinarily small or large that they make no impression on us. We pass them as we do great works of art which we do not understand—the achievements of the geniuses are, under such circumstances, merely so many more pictures and statues! The use of such words as "immeasurably," "extraordinarily," "immensely," and a multiplicity of others, is meaningless. They do not call to our minds anything with which we may compare the magnitudes in question, and often serve to add a nebulous cloud of mystery.

Consider, for example, the following quantities:

1. The distance between the earth and moon = 240,000 miles;
2. The distance between earth and sun = 93,000,000 miles;
3. One gram of electrons contains 10^{27} electrons;
4. The velocity of light in vacuo = 3×10^{10} cms/sec; or 186,000 mi/sec.;

5. The radius of the electron $=2x10^{-13}$ cm.;

6. The mass of one electron $=9x10^{-28}$ gm.;

7. The diameter of Betelgeuse $=270,000,000$ miles;

8. The distance between Sirius and the Earth $=47x10^{12}$ miles and

9. The radius of Einstein's universe $=10^{26}$ cms.

We do not have even a vague idea of these magnitudes.

1. Thus, consider the distance between the earth and the moon. An airplane, traveling continuously at the rate of 100 miles per hour, would require 100 days to make the journey from earth to moon, or roughly 3.5 months. A train traveling continuously at the rate of 60 miles per hour would require about 167 days or approximately 5.5 months to make same journey.

2. The distance between the sun and earth is roughly 93, 000,000 miles. To get an idea how large this distance is, let us assume that an airplane, traveling continuously at the rate of 200 miles per hour, is to make the journey from the earth to the sun. It would take the airplane approximately 53 years to do this. Another idea of this distance may be had by assuming that bodies of the size of the earth are placed in a straight line extending from the earth to the sun. The number of such bodies required would be 10,000. Still another idea of this distance may be gained from the fact that, although light travels at the rate of 186,000 miles per sec., it takes light about 8.4 minutes to come from the sun to us.

3. The number of electrons in a gram of electrons is about 10^{27}. To get an idea of this magnitude, suppose that this number of electrons were placed in a straight line, side by side; then, the length of this line would be sufficient to extend from the earth to the sun and back 14 times! It would also be sufficiently long to extend from the Earth to Neptune!

4. The velocity of light is approximately $3x10^{10}$ cms/sec, or 186,000 miles per sec. The circumference of the earth is about 25,000 miles. A person traveling continuously with the speed of light could go round the earth 7 times before he would hear the last syllable of his friend's wish "Good luck to you". A society lady, "taking off" at noon, and traveling continuously with the speed of light could have afternoon tea on Neptune (Earth's time). Again, a person traveling with the speed of

light could make the round trip from Boston to San Francisco more than 20 times in the time it would take him to say "Good morning".

5. To get an idea of how small an electron is, it is only necessary to reflect that the number of electrons that could be placed on a man's finger nail, if placed in a row, side by side, would extend from the earth to the sun; or, if we assume each electron to be 1 cm. in dia., the row of electrons would be equal to the radius of Einstein's universe. Again, if each person in the U. S. had as many pennies as the number of electrons that could be placed on a man's finger nail, he would have more money than all the nations combined!

6. The mass of the electron is so small that it takes about 10^{27} of them to make one gram, or about 500 times this number to make one pound. In this connection, it may be mentioned that even molecules are so small that, if we "squeezed" all the "space" out of a 175-pound man and packed the "stuff" of which he is composed as closely as possible, it would occupy less space than that occupied by a pea. From this point of view, a man is largely "empty space". Another idea of the size of molecules may be got by reflecting that, if the earth be assumed to consist of water only, and the molecules of a glass of water (these molecules having been labelled for the purpose of identification) were uniformly distributed throughout the earth, then, on going to the faucet to refill the glass with water, we would find at least one of the original molecules.

7. The diameter of Betelgeuse is approximately 27×10^7 mi. This star is so large that it will contain a million of our suns. A beam of light wil require about 23 minutes to traverse the dia. of this star. If bodies of the size of the earth were placed in a row, side by side, it would require about 34,000 such bodies to reach from one end of a diameter of Betelgeuse to the other, or about 340 bodies of the size of our sun.

8. The distance between Sirius and the Earth is 47×10^{12} mi. Light, traveling at the rate of 186,000 miles per sec., takes about 8.4 years to come to the earth. It would require about 170,000 stars of the size of Betelgeuse to reach from the earth to Sirius, assuming, of course, that these stars were all placed in a row.